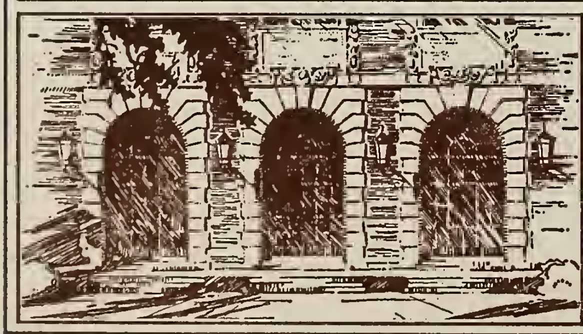

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A NOTE ON THE NON-EXISTENCE OF MULTIVALUE A-STABLE METHODS OF ORDER GREATER THAN TWO

by

C. W. Gear

March 1973

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* Supported in part by contract US AEC AT(11-1)1469.

ABSTRACT

It is shown that the Dahlquist result limiting the order of A-stable multistep methods also applies to multivalue methods.

INTRODUCTION

In 1956, Dahlquist [1] showed that the maximum order of a strongly stable k-step linear multistep method is k+1. In 1963, he showed [2] that the maximum order of an A-stable multistep method is two. In 1966, the author [3] showed that an extension of multistep methods, called multivalue methods, led to a larger class of methods, called modified k-step methods, such that the order of a stable method could be increased to 2k. Since then, the question of whether a multivalue method of order greater than two could be A-stable has remained. The answer is no and is shown by reducing the problem to the problem studied by Dahlquist.

PROOF OF RESULT

Consider the test equation $y' - \lambda y = 0$. Let $\underline{a} = [a_0, a_1, \ldots a_k]^T = [y, hy', \ldots h^k y^{(k)}/k!]^T$ be the data saved in the normal form of a multivalue method, and let $F(\underline{a}) = \underline{\delta}_1^T \, \underline{a} - \mu \, \underline{\delta}_0^T \, \underline{a} = a_1 - \mu a_0$ where $\mu = h\lambda$. Then, from equation (11.21) in reference [4], the multivalue method is

$$\underline{a}_n = A\underline{a}_{n-1} - \underline{\ell}[\frac{\partial F}{\partial \underline{a}}\underline{\ell}]^{-1} F(A\underline{a}_{n-1})$$

$$= A\underline{a}_{n-1} - \underline{\ell}[\underline{\delta}_1^T \underline{\ell} - \mu\, \underline{\delta}_0^T \underline{\ell}]^{-1} (\underline{\delta}_1^T - \mu\, \underline{\delta}_0^T) A\underline{a}_{n-1}$$

$$= \left[I - \frac{\underline{\ell}(\underline{\delta}_1^T - \mu\, \underline{\delta}_0^T)}{\ell_1 - \mu\ell_0} \right] A\underline{a}_{n-1}$$

$$= S\underline{a}_{n-1}$$

where $\underline{\ell} = [\ell_0, \ell_1, \ldots]^T$.

If the method is A-stable, then $\underline{a}_n = S^n \underline{a}_0 \to 0$ as $n \to \infty$ for fixed h whenever $\mathrm{Re}(\mu) < 0$. Hence, A-stability implies that all eigenvalues of S are inside the unit circle when $\mathrm{Re}(\mu) < 0$.

The eigenvalues of S are the same as the eigenvalues of

$$A\left[I - \underline{\ell}\frac{(\underline{\delta}_1^T - \mu\, \underline{\delta}_0^T)}{\ell_1 - \mu\ell_0} \right]$$

Let $A - \xi I = [\underline{c}_0, \underline{c}_1, \ldots, \underline{c}_k]$ where $\underline{c}_i$ is the i-th column of $A - \xi I$, and let $A\underline{\ell} = \underline{v}$. Then we want to study the solutions of

$$0 = \det\left[A - \underline{v}\frac{\underline{\delta}_1^T - \mu\, \underline{\delta}_0^T}{\ell_1 - \mu\ell_0} - \xi I \right]$$

$$= \det[\underline{c}_0 + \underline{v}\frac{\mu}{\ell_1 - \mu\ell_0}, \underline{c}_1 - \underline{v}\frac{1}{\ell_1 - \mu\ell_0}, \underline{c}_2, \ldots, \underline{c}_k]$$

$$= \det[\underline{c}_0, \underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k]$$

$$+ \det[\underline{v}\frac{\mu}{\ell_1 - \mu\ell_0}, \underline{c}_1, c_2, \ldots, \underline{c}_k]$$

$$- \det[\underline{c}_0, \underline{v}\frac{1}{\ell_1 - \mu\ell_0}, \underline{c}_2, \ldots, \underline{c}_k]$$

$$= \frac{1}{\ell_1 - \mu\ell_0}\left[\ell_1\{\det(A-\xi I) - \det[c_0, \frac{\underline{v}}{\ell_1}, \underline{c}_2, \ldots, \underline{c}_k]\}\right.$$

$$\left. - \mu\ell_0\{\det(A-\xi I) - \det[\frac{\underline{v}}{\ell_0}, \underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k]\}\right]$$

$$= \frac{1}{\ell_1 - \mu\ell_0}\ [\rho(\xi) - \mu\sigma(\xi)] \qquad (1)$$

where $\rho(\xi)$ and $\sigma(\xi)$ are the weighted sums of some determinants which are polynomials in ξ and independent of μ. Now this is the same as the polynomial that determines the roots of the constant coefficient difference equation arising when a linear multistep method is applied to $y' = \lambda y$. If the method has order p, then one eigenvalue of S, namely one zero of (1) must approximate e^{μ} to order p, and this is the necessary and sufficient condition for ρ and σ to determine a p-th order multistep method. If the eigenvalues of S are less than one for all $\mathrm{Re}(\mu) < 0$, then the corresponding multistep method is A-stable, but this is not possible if $p > 2$.

LIST OF REFERENCES

[1] Dahlquist, G., "Numerical Integration of Ordinary Differential Equations," Math. Scandinavica, 4,(1956) pp. 33-50.

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It is shown that the Dahlquist result limiting the order of A-stable multistep methods also applies to multivalue methods.

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